



Получена: 01.04.2024 г.

Приета: 22.04.2024 г.

GENERALIZED THEORETICAL STRESS-STRAIN RELATIONSHIP IN SOLIDS WITH MATERIAL NONLINEARITY IN THE CASE OF SMALL DEFORMATIONS

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Keywords: material nonlinearity, stress-strain relation, hyperbolic functions, small deformations, strain energy

ABSTRACT

An original theoretical model is presented to formulate a variety of material nonlinearities in ductile, brittle and hyperelastic solids in the case of small deformations. The various stress-strain relationships are described in a generalized but very realistic, clear and compact way. The representation of the analytical formula for hyperbolic tangent by means of exponents allows with only one parameter for the stresses and only a few parameters for the strains to describe the characteristic stress-strain curves for the steels, the different curves in compression and tensile strength for brittle materials such, as a concrete, as well as the specific curves for hyperelastic materials. These parameters are the natural values of the maximum stresses and ultimate strains that occur in solids under static loading.

1. Introduction

Realistic modeling and correct prediction of the natural physically nonlinear behavior of various types of materials is an extremely important part of both theoretical and applied or computational mechanics. This branch of physics has been developed for more than a century and has gained great momentum in recent decades. The theory of linear elasticity does not give a correct picture of the real mechanical behavior of the solids, since it does not take into account the different degrees of failure, nor does it give the limit values of the forces that a structure can withstand before it loses its capacity according to one or another criterion. In

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order to find answers to these questions, dozens of material nonlinearity models describing various types of plastic, brittle, composite and other types of inelastic materials have been developed, e.g. [1 – 32], many mathematical methods have been applied to solve the system of nonlinear equations and a number of computer programs have been designed. In addition to the huge variety of physical models for the material nonlinearity in solids (and soils), which are not very easy to use in practice, a basic challenge is to ensure the convergence of the problem. For many cases used standard monolithic or modified Newton-Raphson schemes and some quasi-static methods used fail. Moreover, some of the used more accurate models for material nonlinearity solid behavior and the successfully applied computational methods for them, especially for larger scale problems, involve large computational effort and considerable computational time, e.g. [33], etc.

To overcome these difficulties, the aim is to approximate the nonlinear behavior of different types of materials as quasi-linear, where, and as far as possible. This would greatly simplify both the formulation and the solution of the system of equations. On the other hand, the simple but natural modeling of different types of material nonlinearity and the solution of the nonlinear problem by the standard finite element method, based entirely on the behavior of a linear-elastic material, would greatly extend the practical application of the nonlinear mechanics. However, its use is often insufficiently understood by structural engineering students [34] and structural engineers due to the excessive complexity of the theoretical models and the not insignificant difficulties in obtaining an easily applicable practical result. With the desire to find simultaneously a sufficiently simple, physically correct, and very efficient solution to the problem of material nonlinearity in solids, this paper proposes a logically consistent generalized model of the stress-strain relationship in the case of small deformations.

2. Basic theoretical stress-strain relationship

There are several theoretical and theoretical-experimental models, as well as their modifications, in which some form of hyperbolic dependence is used to describe the nonlinear relationship between stresses and strains for different types of materials, e.g. [10] for representing the elastic-plastic mechanical behavior of soils, [12] for lead-free solder joints, or [13] for nonlinear dynamic rupture processes in diffuse fracture zones, etc.

The proposed theoretical relationship between stresses and strains is not based on any of these. It is not specific to any particular type of material. On the other hand, no matter how well conditioned a theoretical stress-strain relationship is, when it is possible to obtain experimentally a nonlinear stress-strain relationship and it can be used in the calculations, it is preferable to do so. Such relationships are available in large quantities, such as the engineering stress-strain curves, e.g. [35 – 37], also in many computer programs, etc. The proposed theoretical stress-strain relationship is developed to describe in principle way, within the framework of infinitesimal strain theory and Cauchy stress tensor, the model of material nonlinearity of solids and its use in the computational methodology in the standard finite element method. This theoretical formulation allows a wide range of stress-strain relationships to be realistically described for materials ranging from ductile to brittle or hyperelastic material, with an appropriate choice of the model constant parameters.

The basic stress-strain relationship or material constitutive equation is defined by the following formula (Fig. 1):

$$\sigma(\varepsilon) = \sigma_{lim} \cdot \tanh\left(\frac{\varepsilon}{\varepsilon_{rul}}\right). \quad (1)$$

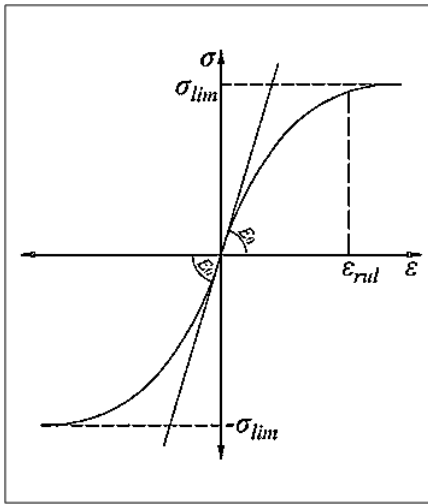


Figure 1. Stress-strain relationship

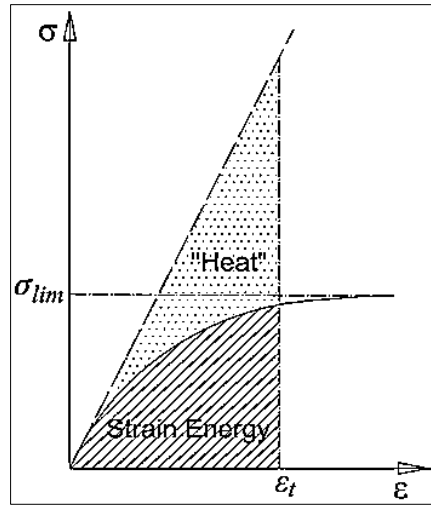


Figure 2. Energy distribution in material nonlinearity

In formula (1) $\sigma(\varepsilon)$ is the stress in the elementary volume, defined by the Cauchy stress tensor, and σ_{lim} is its limit value to which it tends to but does not reach; " ε " is the strain according to infinitesimal strain theory, defined by the Cauchy's strain tensor, and ε_{rul} is the governing strain that can be considered as the limit value at which failure occurs. Since for any material no strictly defined values can be fixed for the stresses and or strains at which its failure begins or ends, and these are always a matter of assumption and some chosen criteria, therefore σ_{lim} and ε_{rul} are close to different real limits of stresses and strains but they are not exactly some of them. The parameter ε_{rul} can be taken as the limit strain in the cases when its numerical values satisfy the corresponding conditions for the magnitude of the real limit strain when the solid is completely destroyed. From a practical point of view, in order to obtain a "good" stress-strain curve, another value of ε_{rul} can be defined, independent of the limit strain, smaller or larger. Analogous considerations can also be used to choose the limit stress. For any stress-strain relationship σ_{lim} – the limit stress value – and ε_{rul} – the governing strain – are constant parameters.

In the case where the stress is defined by formula (1), the strain will be given by the expression:

$$\varepsilon = \varepsilon_{rul} \cdot \operatorname{artanh} \left[\frac{\sigma(\varepsilon)}{\sigma_{lim}} \right] = \frac{1}{2} \varepsilon_{rul} \cdot \ln \left[\frac{\sigma_{lim} + \sigma(\varepsilon)}{\sigma_{lim} - \sigma(\varepsilon)} \right]. \quad (2)$$

The modulus of nonlinear deformation of the material is given by the expression:

$$\frac{\partial \sigma(\varepsilon)}{\partial \varepsilon} = \frac{\sigma_{lim}}{\varepsilon_{rul}} \cdot \left[1 - \tanh^2 \left(\frac{\varepsilon}{\varepsilon_{rul}} \right) \right] = \frac{E_0}{\cosh^2 \left(\frac{\varepsilon}{\varepsilon_{rul}} \right)} = E(\varepsilon). \quad (3)$$

In the expression (3) the ratio $\frac{\sigma_{lim}}{\varepsilon_{rul}} = E_0$ is the initial value of the modulus of (nonlinear) deformation at $\varepsilon = 0$ or the "initial" tangent line of the σ - ε curve at the point $\varepsilon = 0$.

In fact, it is the modulus of elasticity or Young's modulus of the linear elastic solid (see Fig. 1 and Fig. 2). The modulus $E(\varepsilon)$ (Fig. 3) is the slope of the stress-strain curve at any point and it is the tangent modulus [38] (Fig. 4), which is essentially the modulus of the "purely" elastic part of the strains of the "conditionally unbroken" part of the solid. This is also a materials property. It could be accepted as "nonlinear" modulus of the "residual" elasticity or modulus of the nonlinear deformation. " σ " and " ε " are used throughout this exposition to refer to both linear and transverse stresses and strains, respectively. In Fig. 3 the continuous line represents the (nonlinear) modulus $E(\varepsilon)$ and the dashed line represents the initial value of the modulus

$E_0 = \frac{\sigma_{lim}}{\varepsilon_{rul}}$. In Fig. 3 and Fig. 4 for the stress-strain relation formula (1) is used with

$\sigma_{lim} = 26 \text{ MPa}$ and $\varepsilon_{rul} = 0,00078$.

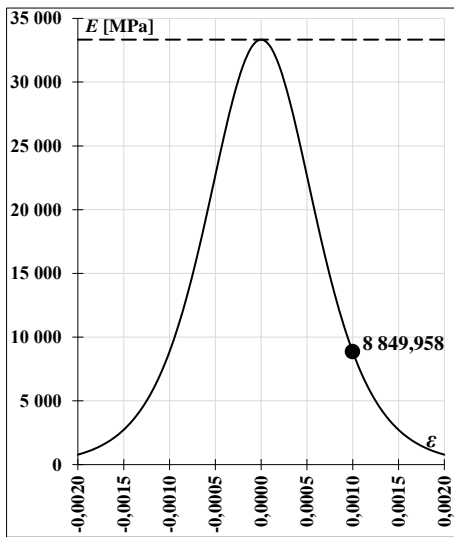


Figure 3. Nonlinear modulus $E(\varepsilon)$ and linear modulus E_0

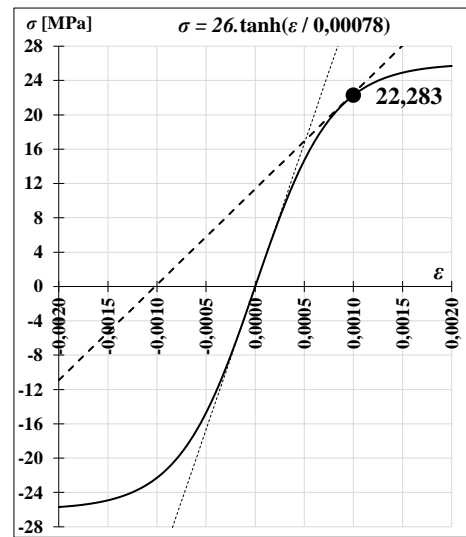


Figure 4. Tangent moduli for $\varepsilon = 0,001$ and $\varepsilon = 0$

3. Stresses and strains in case of unloading

As it is known, the hatched part of the diagram in Fig. 2, between the function $\sigma(\varepsilon)$ and the abscissa " ε ", represents the strain energy density – the potential energy per unit volume stored in him during the development of nonlinear deformations for the corresponding value of " ε ". The area of the diagram between the function $\sigma(\varepsilon)$, a straight line parallel to the abscissa (e.g., $\sigma = \sigma_{lim}$ in Fig. 2) and the tangent to it at the origin – the line $\sigma_{elast} = E_0 \cdot \varepsilon$ – gives the energy dissipated in the partial destruction of the body, in general – heat. The difference between the stresses in the perfectly elastic solid and the stresses in real solids can be considered as defining and "corresponding" to the irreversible "residual" or "plastic" strains in the solids.

The use of the term "strain energy potential" may not be the entirely correct terminology for the physical behavior of solids in material nonlinearity solid mechanics, but there are sufficient reasons, both from a physical and computational point of view, to adopt and use it. It is undeniable that solids retain "some" of their initial elastic properties virtually throughout the

failure process. The strength and the integrity of the deformable solid are partially degraded due to the external load and the modulus of elasticity is reduced, but it still has enough of the properties of an "elastic" solid. This allows us to assume that "some part" of the solid still has "pure" elastic properties. Considering this, and taking into account the nonlinearity of the material, it can be assumed that the solid has a strain energy potential.

It is generally accepted that the strains and also the correlations of the strains can be decomposed into an elastic and a plastic part. This is the basic principle of elastoplasticity (linear elastic-perfectly plastic behavior). Even though it is not quite correct to divide strains into "purely" elastic and "purely" plastic, there are enough reasons to do so for practical purposes. This division is more or less conditional, since the "plastic" strains can be different depending on the type of material nonlinearity (the exception is the case represented by the ideal elastic-plastic stress-strain diagram, but it is idealized for the behavior of the whole specimen under study, not for an elementary volume, section or finite element). For example, such a partition for different types of materials is used in [28, 29]. A similar way to describe the deformation behavior of the concrete in practice is to use *the elasticity coefficient* v' – the ratio of the elastic part of the deformation to the sum of the elastic and plastic deformations, i.e.: $v' = \varepsilon_{el}/(\varepsilon_{el} + \varepsilon_{pl})$ and *the plasticity coefficient* λ – the ratio of the plastic part of the strains to the sum of the elastic and plastic deformations, i.e.: $\lambda = \varepsilon_{pl}/(\varepsilon_{el} + \varepsilon_{pl})$ [27, 39].

With regard to the above-mentioned conditional "pure" separation of strains into elastic and plastic parts, and considering formula (3) and the properties of hyperbolic functions, in the case of the σ - ε relation defined by formula (1), the following can be written:

$$\varepsilon_{pl} = \varepsilon - \varepsilon_{el} = \left(1 - \frac{E(\varepsilon)}{E_0}\right) \cdot \varepsilon = \tanh^2\left(\frac{\varepsilon}{\varepsilon_{rul}}\right) \cdot \varepsilon. \quad (4)$$

In formula (4), the following assumption was made for the "purely" elastic part of the strains:

$$\frac{\varepsilon_{el}}{\varepsilon} = \frac{E(\varepsilon)}{E_0} \rightarrow \varepsilon_{el} = \frac{E(\varepsilon)}{E_0} \cdot \varepsilon = \frac{\varepsilon}{\cosh^2\left(\frac{\varepsilon}{\varepsilon_{rul}}\right)}. \quad (5)$$

In Fig. 5, the continuous line represents the "purely" elastic part of the strains, the dashed line represents the "purely" plastic part of the strains, and the dotted line represents the total strain. The assumption (5) is based on the hypothesis that the plastic part of the strains ε_{pl} refers to the total strains " ε " as the "residual" part of (nonlinear) modulus $E(\varepsilon)$ refers to the initial value of the modulus E_0 .

Then for the stress we obtain (Fig. 7):

$$\sigma(\varepsilon)_{pl} = \sigma_{el} - \sigma(\varepsilon) = E_0 \cdot \varepsilon - \sigma_{lim} \cdot \tanh\left(\frac{\varepsilon}{\varepsilon_{rul}}\right). \quad (6)$$

In Fig. 6, the continuous line represents the nonlinear stress, which is the "residual" elastic stress defined by formula (1); the dotted line represents the stress in the solid as a perfectly elastic body, and the dashed line represents the difference between them. This difference could be considered as a fictive "plastic" stress, corresponding to the dissipated bond energy in the body during the destruction of solids with material nonlinearity due to some kind of [monotonic \(static\)](#) loading. Usually, this dissipated energy takes the form of heat.

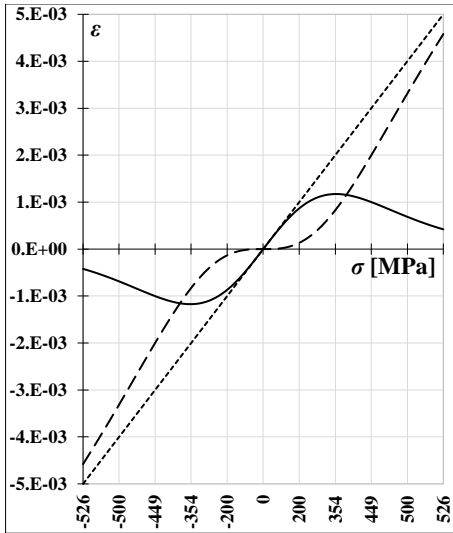


Figure 5. Curves of the elastic and plastic strains

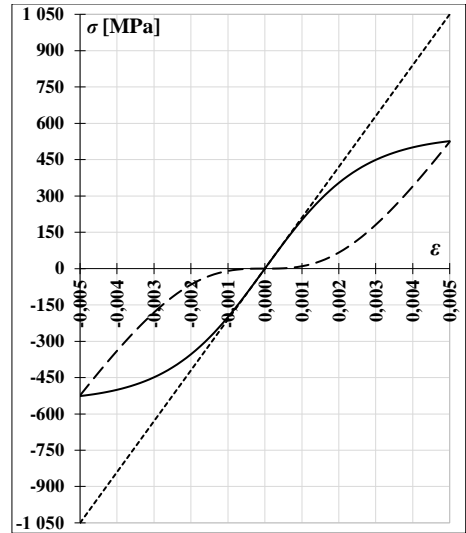


Figure 6. Elastic and "plastic" stress curves

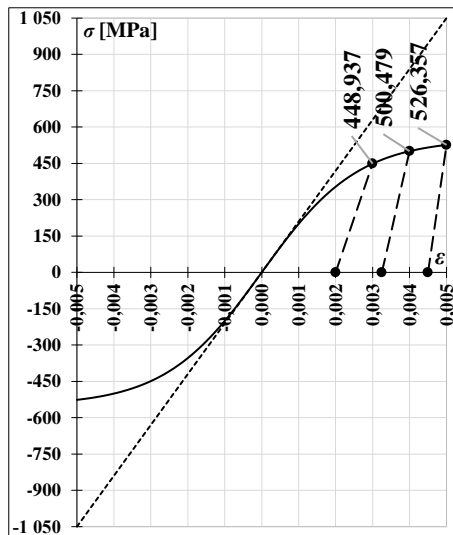


Figure 7. Plastic and elastic strains in case of unloading

It is very important to define the line of unloading of the stress, especially for the cases with cyclic loading, such as the seismic impact. In this case, the modulus of elasticity with which this unloading takes place is given by the expression (Fig. 9):

$$E(\varepsilon)_{unld} = \frac{\sigma(\varepsilon)}{\varepsilon_{el}}. \quad (7)$$

In formula (6) $\sigma(\varepsilon)$ is calculated by formula (1) and ε_{el} is calculated by formula (5). In Fig. 7, the continuous line represents the nonlinear stress, which is the "residual" elastic stress defined by formula (1); the dotted line represents the stress in the solid as a perfectly elastic body, and

the dashed lines are the lines of unloading at three arbitrarily chosen points. The point where each of these stress relief lines crosses the abscissa ("ε" axis) corresponds to the irreversible plastic deformation that occurs at the stress corresponding to the intersection of the continuous nonlinear stress line $\sigma(\varepsilon)$ with the corresponding unloading line. As can be seen from Fig. 7, the fraction of "irreversible" plastic strains in the total strains gradually increases with load increasing and hence stress at the expense of elastic strains.

To represent the curves and lines in Fig. 5 to Fig. 7, formulas (1) to (7) are used, with $\sigma_{lim} = 550,0$ MPa and $\varepsilon_{rul} = 0,0026$. In Fig. 7 stress $\sigma(\varepsilon) = 448,937$ MPa is obtained by formula (1) for $\varepsilon = 0,003$, $\varepsilon_{pl} = 0,001999$ is obtained by formula (4) and $\varepsilon_{el} = 0,001001$ is obtained by formula (5); stress $\sigma(\varepsilon) = 500,479$ MPa is obtained by formula (1) for $\varepsilon = 0,004$, $\varepsilon_{pl} = 0,003312$ is obtained by formula (4) and $\varepsilon_{el} = 0,000688$ is obtained by formula (5); stress $\sigma(\varepsilon) = 500,479$ MPa is obtained by formula (1) for $\varepsilon = 0,005$, $\varepsilon_{pl} = 0,004570$ is obtained by formula (4) and $\varepsilon_{el} = 0,000421$ is obtained by formula (5).

4. Transformation of the governing equations of solid mechanics for proposed model of material nonlinearity

As is well known, the governing equations of linear elasticity with respect to a rectangular Cartesian coordinate system in engineering notation, here and throughout the paper to the end, have the following form, e.g. [1, 40, 41], etc.:

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho \cdot g_x &= \rho \cdot \frac{\partial^2 u}{\partial t^2}, \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho \cdot g_y &= \rho \cdot \frac{\partial^2 v}{\partial t^2}, \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \rho \cdot g_z &= \rho \cdot \frac{\partial^2 w}{\partial t^2}. \end{aligned} \quad (8)$$

In the case of only static load and without considering the mass of the body, the equations (8) take the simple form:

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} &= 0, \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} &= 0, \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} &= 0. \end{aligned} \quad (9)$$

If the stress-strain relation (1) holds for each component of the Cauchy stress tensor σ_{ij} ("i" and "j" $\equiv x, y, z$), considering (3), we obtain:

$$\begin{aligned}
& E_0 \cdot \left[1 - \tanh^2 \left(\frac{\varepsilon_{xx}}{\varepsilon_{rul}} \right) \right] \cdot \frac{\partial \varepsilon_{xx}}{\partial x} + E_0 \cdot \left[1 - \tanh^2 \left(\frac{\varepsilon_{yx}}{\varepsilon_{rul}} \right) \right] \cdot \frac{\partial \varepsilon_{yx}}{\partial y} + E_0 \cdot \left[1 - \tanh^2 \left(\frac{\varepsilon_{zx}}{\varepsilon_{rul}} \right) \right] \cdot \frac{\partial \varepsilon_{zx}}{\partial z} = \\
& E_0 \cdot \left[1 - \tanh^2 \left(\frac{\varepsilon_{xy}}{\varepsilon_{rul}} \right) \right] \cdot \frac{\partial \varepsilon_{xy}}{\partial x} + E_0 \cdot \left[1 - \tanh^2 \left(\frac{\varepsilon_{yy}}{\varepsilon_{rul}} \right) \right] \cdot \frac{\partial \varepsilon_{yy}}{\partial y} + E_0 \cdot \left[1 - \tanh^2 \left(\frac{\varepsilon_{zy}}{\varepsilon_{rul}} \right) \right] \cdot \frac{\partial \varepsilon_{zy}}{\partial z} = \\
& E_0 \cdot \left[1 - \tanh^2 \left(\frac{\varepsilon_{xz}}{\varepsilon_{rul}} \right) \right] \cdot \frac{\partial \varepsilon_{xz}}{\partial x} + E_0 \cdot \left[1 - \tanh^2 \left(\frac{\varepsilon_{yz}}{\varepsilon_{rul}} \right) \right] \cdot \frac{\partial \varepsilon_{yz}}{\partial y} + E_0 \cdot \left[1 - \tanh^2 \left(\frac{\varepsilon_{zz}}{\varepsilon_{rul}} \right) \right] \cdot \frac{\partial \varepsilon_{zz}}{\partial z} = \\
& = \frac{E_0}{\cosh^2 \left(\frac{\varepsilon_{xx}}{\varepsilon_{rul}} \right)} \frac{\partial \varepsilon_{xx}}{\partial x} + \frac{E_0}{\cosh^2 \left(\frac{\varepsilon_{yx}}{\varepsilon_{rul}} \right)} \frac{\partial \varepsilon_{yx}}{\partial y} + \frac{E_0}{\cosh^2 \left(\frac{\varepsilon_{zx}}{\varepsilon_{rul}} \right)} \frac{\partial \varepsilon_{zx}}{\partial z} = \\
& = \frac{E_0}{\cosh^2 \left(\frac{\varepsilon_{xy}}{\varepsilon_{rul}} \right)} \frac{\partial \varepsilon_{xy}}{\partial x} + \frac{E_0}{\cosh^2 \left(\frac{\varepsilon_{yy}}{\varepsilon_{rul}} \right)} \frac{\partial \varepsilon_{yy}}{\partial y} + \frac{E_0}{\cosh^2 \left(\frac{\varepsilon_{zy}}{\varepsilon_{rul}} \right)} \frac{\partial \varepsilon_{zy}}{\partial z} = \\
& = \frac{E_0}{\cosh^2 \left(\frac{\varepsilon_{xz}}{\varepsilon_{rul}} \right)} \frac{\partial \varepsilon_{xz}}{\partial x} + \frac{E_0}{\cosh^2 \left(\frac{\varepsilon_{yz}}{\varepsilon_{rul}} \right)} \frac{\partial \varepsilon_{yz}}{\partial y} + \frac{E_0}{\cosh^2 \left(\frac{\varepsilon_{zz}}{\varepsilon_{rul}} \right)} \frac{\partial \varepsilon_{zz}}{\partial z} = \\
& = E(\varepsilon_{xx})_{xx} \cdot \frac{\partial \varepsilon_{xx}}{\partial x} + E(\varepsilon_{yx})_{yx} \cdot \frac{\partial \varepsilon_{yx}}{\partial y} + E(\varepsilon_{zx})_{zx} \cdot \frac{\partial \varepsilon_{zx}}{\partial z} = 0, \\
& = E(\varepsilon_{xy})_{xy} \cdot \frac{\partial \varepsilon_{xy}}{\partial x} + E(\varepsilon_{yy})_{yy} \cdot \frac{\partial \varepsilon_{yy}}{\partial y} + E(\varepsilon_{zy})_{zy} \cdot \frac{\partial \varepsilon_{zy}}{\partial z} = 0, \\
& = E(\varepsilon_{xz})_{xz} \cdot \frac{\partial \varepsilon_{xz}}{\partial x} + E(\varepsilon_{yz})_{yz} \cdot \frac{\partial \varepsilon_{yz}}{\partial y} + E(\varepsilon_{zz})_{zz} \cdot \frac{\partial \varepsilon_{zz}}{\partial z} = 0.
\end{aligned} \tag{10}$$

Written with respect to the displacements and taking into account their relations to the deformations of the Cauchy's strain tensor: $\varepsilon_{xx} = \frac{\partial u}{\partial x}$, $\varepsilon_{yx} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$, $\varepsilon_{zx} = \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)$, $\varepsilon_{xy} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$, $\varepsilon_{yy} = \frac{\partial v}{\partial y}$, $\varepsilon_{zy} = \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)$, $\varepsilon_{xz} = \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)$, $\varepsilon_{yz} = \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)$, $\varepsilon_{zz} = \frac{\partial w}{\partial z}$, the last part of equations (10) take the form:

$$\begin{aligned}
&= E(\varepsilon_{xx})_{xx} \cdot \frac{\partial^2 u}{\partial x^2} + E(\varepsilon_{yx})_{yx} \cdot \frac{\partial^2 u}{\partial y^2} + E(\varepsilon_{zx})_{zx} \cdot \frac{\partial^2 u}{\partial z^2} = 0, \\
&= E(\varepsilon_{xy})_{xy} \cdot \frac{\partial^2 v}{\partial x^2} + E(\varepsilon_{yy})_{yy} \cdot \frac{\partial^2 v}{\partial y^2} + E(\varepsilon_{zy})_{zy} \cdot \frac{\partial^2 v}{\partial z^2} = 0, \\
&= E(\varepsilon_{xz})_{xz} \cdot \frac{\partial^2 w}{\partial x^2} + E(\varepsilon_{yz})_{yz} \cdot \frac{\partial^2 w}{\partial y^2} + E(\varepsilon_{zz})_{zz} \cdot \frac{\partial^2 w}{\partial z^2} = 0.
\end{aligned} \tag{11}$$

In the first part of the above expression, after revealing the square brackets, the terms $E_0 \cdot \frac{\partial \varepsilon_{ij}}{\partial x_{ij}}$ give that part of the components of the Cauchy's strain tensor that represents the strain energy of the elementary volume as a linear or "perfect" elastic solid. The terms $E_0 \cdot \tanh^2 \left(\frac{\varepsilon_{ij}}{\varepsilon_{rul}} \right) \cdot \frac{\partial \varepsilon_{ij}}{\partial x_{ij}}$ represent that part of the Cauchy's strain tensor components which gives the irreversibly dissipated bond energy of the particles in the destruction process of the elementary volume; this part is usually assumed to be heat. The difference between these two energies, the last two parts of the expression (10), the terms $\frac{E_0}{\cosh^2 \left(\frac{\varepsilon_{ij}}{\varepsilon_{rul}} \right)} \cdot \frac{\partial \varepsilon_{ij}}{\partial x_{ij}} = E(\varepsilon_{ij})_{ij} \cdot \frac{\partial \varepsilon_{ij}}{\partial x_{ij}}$, represent that part of the Cauchy's strain tensor

components that accumulate the strain energy potential of the elastic part of deformations in the case of real "nonlinear" behavior of solids. The elastic energy is due to the still "undamaged" regions of the deformed solid. The indices "ij" denote: the first index "i" denotes the normal of the plane along which the strain is realized and the second index "j" denotes the direction of this strain; "i", "j" and the variable $x \equiv x, y, z$.

Considering the last two parts of expressions (10) and (11), it can be said that the use of dependence (1) as a stress-strain relation reduces the solution of the problem for a material nonlinear solid under static load to the determine of the deformed and stressed state of a pseudo-orthotropic quasi-linear elastic solid, whose anisotropy changes as a result of the loading with respect to all six independent components of the symmetric small strain tensor, the Cauchy's strain tensor, and the Cauchy stress tensor, respectively.

In view of the above and the commonly accepted way of expressing the stress-strain relations for orthotropic materials in terms of orthotropic moduli of elasticity and orthotropic Poisson's ratios, e.g. [42], for the pseudo-orthotropic quasi-linear elastic solid that simulates the nonlinear material behavior defined by formula (1), taking into account expressions (10) and (11), one could accept that the Hooke's law in three dimensions take the form:

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E(\varepsilon_{xx})_{xx}} \cdot \sigma_x - \frac{\nu}{E(\varepsilon_{yy})_{yy}} \cdot \sigma_y - \frac{\nu}{E(\varepsilon_{zz})_{zz}} \cdot \sigma_z; \varepsilon_{yx} = \frac{1}{E(\varepsilon_{yx})_{yx}} \cdot \tau_{yx} = \varepsilon_{xy} = \frac{1}{E(\varepsilon_{xy})_{xy}} \cdot \tau_{xy}; \\
\varepsilon_{yy} &= \frac{\nu}{E(\varepsilon_{xx})_{xx}} \cdot \sigma_x - \frac{1}{E(\varepsilon_{yy})_{yy}} \cdot \sigma_y - \frac{\nu}{E(\varepsilon_{zz})_{zz}} \cdot \sigma_z; \varepsilon_{zy} = \frac{1}{E(\varepsilon_{zy})_{zy}} \cdot \tau_{zy} = \varepsilon_{yz} = \frac{1}{E(\varepsilon_{yz})_{yz}} \cdot \tau_{yz}; \\
\varepsilon_{zz} &= \frac{\nu}{E(\varepsilon_{xx})_{xx}} \cdot \sigma_x - \frac{\nu}{E(\varepsilon_{yy})_{yy}} \cdot \sigma_y - \frac{1}{E(\varepsilon_{zz})_{zz}} \cdot \sigma_z; \varepsilon_{xz} = \frac{1}{E(\varepsilon_{xz})_{xz}} \cdot \tau_{xz} = \varepsilon_{zx} = \frac{1}{E(\varepsilon_{zx})_{zx}} \cdot \tau_{zx}.
\end{aligned} \tag{12}$$

5. Developing the basic theoretical relationships between stress and strain for various types of materials

Taking advantage of the possibility to represent \tanh as a quotient of \sinh and \cosh , as well as a combination of exponent degrees, the stress-strain relationship of formula (1) can be developed and generalized using additional appropriately chosen parameters ε_{rul} and σ_{lim} , as some auxiliary constants, to describe various types of material nonlinearity. Both ductile (perfectly plastic) materials (Fig. 8) and brittle (with different tensile and compressive strengths) materials (Fig. 9), various hyperelastic materials (Fig. 10), or materials with different types of plasticity (Fig. 11) can be described. By appropriately varying the values of some of the parameters, hypothetical real or provisional materials can be described (Fig. 12).

For perfectly plastic material, the σ - ε relationship, shown in Fig. 8 is given by formula

$$(1): \sigma(\varepsilon) = \sigma_{lim} \cdot \tanh\left(\frac{\varepsilon}{\varepsilon_{rul}}\right), \text{ with } \sigma_{lim} = 200 \text{ MPa and } \varepsilon_{rul} = 0,00105.$$

For brittle materials with much higher compressive strength than tensile strength, such as concrete (Fig. 9), the σ - ε relationships are given by the following formula:

$$\sigma(\varepsilon) = \sigma_{lim} \frac{a_1 \cdot e^{\frac{\varepsilon - b_1}{\varepsilon_{rul,1}}} - c_1}{a_2 \cdot e^{\frac{\varepsilon - b_2}{\varepsilon_{rul,2}}} + c_2}, \quad (13)$$

where: $\sigma_{lim} = 24,081 \text{ MPa}$, $\varepsilon_{rul,1} = 0,00059569$, $\varepsilon_{rul,2} = 0,00003$, $a_1 = 1$, $a_2 = 1$, $b_1 = 0,0001$, $b_2 = 0,0001$, $c_1 = 0,84545$, $c_2 = 1$ for continuous line; $\sigma_{lim} = 20,0 \text{ MPa}$, $\varepsilon_{rul,1} = 0,000461$, $\varepsilon_{rul,2} = 0,000102$, $a_1 = 1$, $a_2 = 1,2$, $b_1 = 0$, $b_2 = 0$, $c_1 = 1$, $c_2 = 1,2$ for dashed line and $\sigma_{lim} = 25 \text{ MPa}$, $\varepsilon_{rul,1} = 0,000461$, $\varepsilon_{rul,2} = 0,000102$, $a_1 = 1$, $a_2 = 0,5$, $b_1 = 0$, $b_2 = 0$, $c_1 = 1$, $c_2 = 1$ for dotted line.

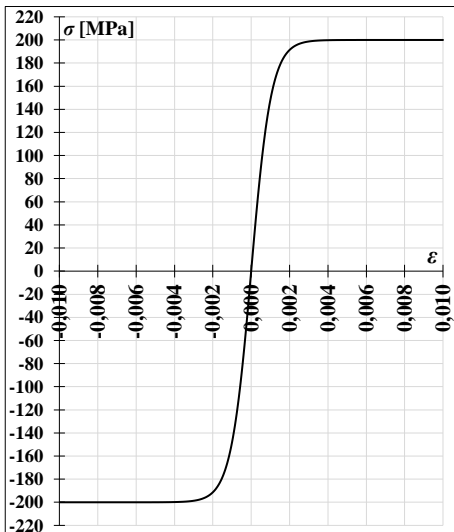


Figure 8. Typical σ - ε curve for plastic material

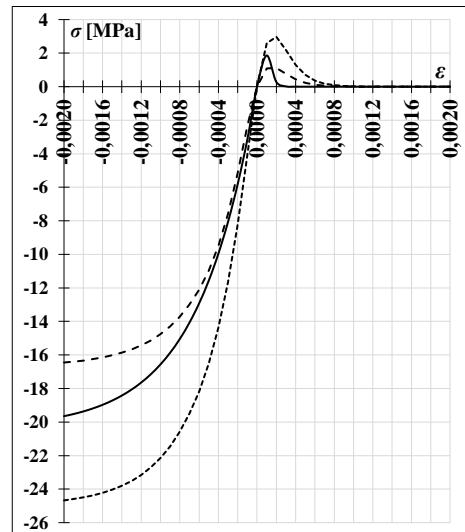


Figure 9. Typical σ - ε curves for brittle materials

In [43], two FEM examples can be seen for a cantilevered concrete and reinforced column subjected to bending and shear, using for the concrete in formula (13) the numerical data for the continuous line shown in Fig. 9, and an example for a stainless-steel beam also subjected to bending and shear, using in formula (1) the values $\sigma_{lim} = 550$ MPa and $\varepsilon_{rul} = 0,002619$ for the steel.

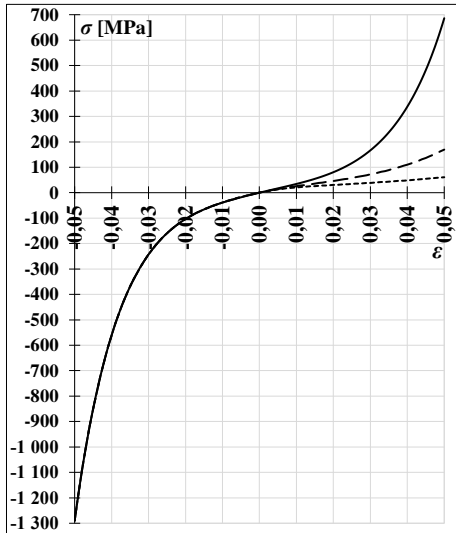


Figure 10. Typical σ - ε curves for different types of hyperelastic materials

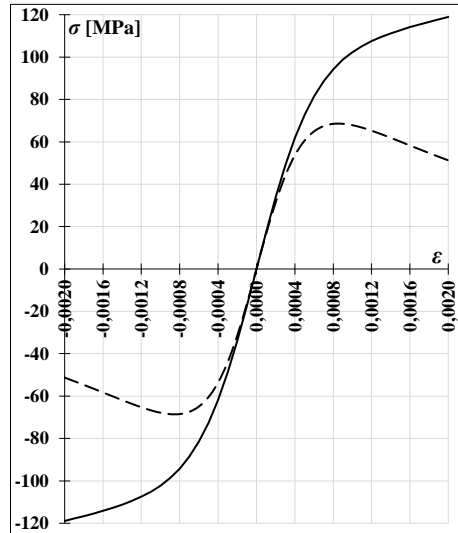


Figure 11. Typical σ - ε curves for some special types of materials

For various types of hyperelastic materials (Fig. 10) the σ - ε curves are obtained by using the following formula:

$$\sigma(\varepsilon) = \sigma_{lim} \frac{\frac{\varepsilon}{e^{\varepsilon_{rul,1}} - e^{\varepsilon_{rul,2}}} - \frac{-\varepsilon}{e^{\varepsilon_{rul,3}} + e^{\varepsilon_{rul,4}}}}{\frac{\varepsilon}{e^{\varepsilon_{rul,3}} + e^{\varepsilon_{rul,4}}}}, \quad (14)$$

where, for the continuous line, $\sigma_{lim} = 20$ MPa, $\varepsilon_{rul,1} = 0,0055$, $\varepsilon_{rul,2} = 0,006$, $\varepsilon_{rul,3} = 0,009$, $\varepsilon_{rul,4} = 0,012$; for the dashed line $\varepsilon_{rul,1} = 0,0065$ and for the dotted line $\varepsilon_{rul,1} = 0,0075$. The values for σ_{lim} , $\varepsilon_{rul,2}$, $\varepsilon_{rul,3}$ and $\varepsilon_{rul,4}$ for the dashed and the dotted lines are the same as for the continuous line.

The dependencies σ - ε dependencies for materials with different types of plasticity, shown in Fig. 11, are given by the following formula:

$$\sigma(\varepsilon) = \sigma_{lim} \cdot \frac{\sinh\left(\frac{\varepsilon}{\varepsilon_{rul,1}}\right)}{\cosh\left(\frac{\varepsilon}{\varepsilon_{rul,2}}\right)}, \quad (15)$$

where $\sigma_{lim} = 100$ MPa for both continuous and dashed lines, but $\varepsilon_{rul,1} = 0,00057$ and $\varepsilon_{rul,2} = 0,0006$ for the dashed line and $\varepsilon_{rul,1} = 0,0006$ and $\varepsilon_{rul,2} = 0,0005$ for the dotted line.

The σ - ε dependence can be written in the following more general form using a few auxiliary constants:

$$\sigma = \sigma_0 + \sigma_{lim} \cdot \frac{a_1 \cdot e^{\frac{\varepsilon - \varepsilon_1}{\varepsilon_{rul,1}}} - a_2 \cdot e^{\frac{-\varepsilon - \varepsilon_2}{\varepsilon_{rul,2}}}}{a_3 \cdot e^{\frac{\varepsilon - \varepsilon_3}{\varepsilon_{rul,3}}} + a_4 \cdot e^{\frac{-\varepsilon - \varepsilon_4}{\varepsilon_{rul,4}}}}, \quad (16)$$

where the ruling deformations $\varepsilon_{rul,1}$, $\varepsilon_{rul,2}$, $\varepsilon_{rul,3}$ and $\varepsilon_{rul,4}$ are the same parameters as in formula (14); a_1 and a_2 are the same scalar parameters as in formula (13) and a_3 and a_4 are analogous parameters to parameters a_1 and a_2 , respectively; $\varepsilon_1 \equiv b_1$, $\varepsilon_2 \equiv b_2$, ε_3 and ε_4 are additional auxiliary modeling parameters for the strains, the same as in formula (13); σ_{lim} is the limiting stress and σ_0 is an initial stress for the corresponding component of the Cauchy stress tensor. For a more general formulation of the σ - ε relationship the scalar parameters c_1 and c_2 in formula (13) could also be added to the numerator and the denominator of formula (16), respectively. This would lead to the inclusion of the initial strains in the σ - ε relationship as well.

Only for illustration of formula (13) and not related to any specified material in Fig. 12 and Fig. 13, $\sigma(\varepsilon)$ and $E(\varepsilon)$ are shown as: σ_{lim} , $\varepsilon_{rul,1}$, $\varepsilon_{rul,2}$, $\varepsilon_{rul,3}$ and $\varepsilon_{rul,4}$ are as in formula (14), $a_1 = a_2 = a_4 = 1$, $a_3 = 5$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$, $\varepsilon_4 = 0,05$, $\sigma_0 = 25,0$ MPa.

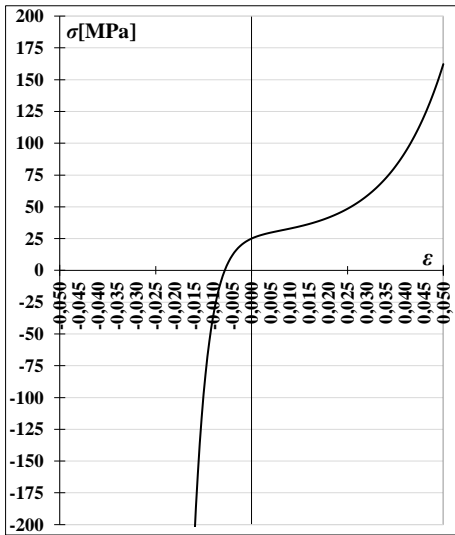


Figure 12. σ - ε curve for hypothetical material

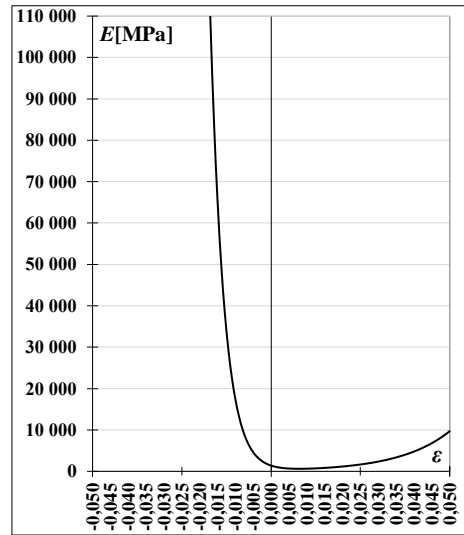


Figure 13. Modulus $E(\varepsilon)$ for hypothetical material

Finally, in order to assess the practical suitability of the proposed methodology, the σ - ε relationship for concrete C 30/37 set by formula (1) with $\sigma_{lim} = 33,0$ MPa and $\varepsilon_{rul} = 0,00119$ is compared with σ - ε relationship according to clause 3.1.7 of EN 1992, EUROCODE 2: Design of concrete structures [44], set by the following formula (in Fig.14, the continuous line is for formula (17)):

$$\sigma_c(\varepsilon_c) = f_{cd} \left[1 - \left(1 - \frac{\varepsilon_c}{\varepsilon_{c2}} \right)^2 \right]; \text{ for } 0 \leq \varepsilon_c \leq \varepsilon_{c2}, \sigma_c(\varepsilon_c) = f_{cd} \text{ for } \varepsilon_{c2} \leq \varepsilon_c \leq \varepsilon_{cu2}, \quad (17)$$

$$E_c = \frac{\partial \sigma_c}{\partial \varepsilon_c} = f_{cd} \frac{2 \cdot (\varepsilon_{c2} - \varepsilon_c)}{\varepsilon_{c2}^2}, \text{ for } 0 \leq \varepsilon_c \leq \varepsilon_{c2}, \quad E_c = 0 \text{ for } \varepsilon_{c2} \leq \varepsilon_c \leq \varepsilon_{cu2}. \quad (18)$$

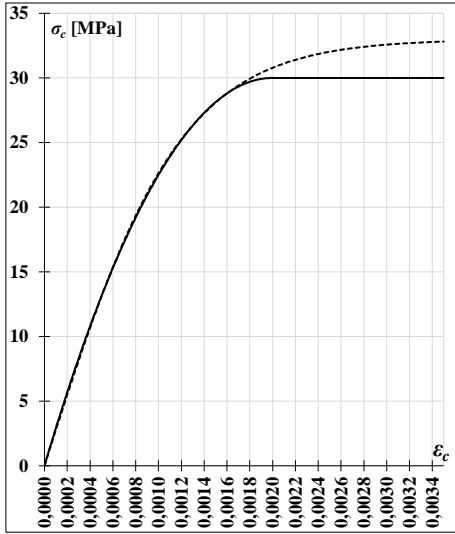


Figure 14. σ - ε curves for concrete

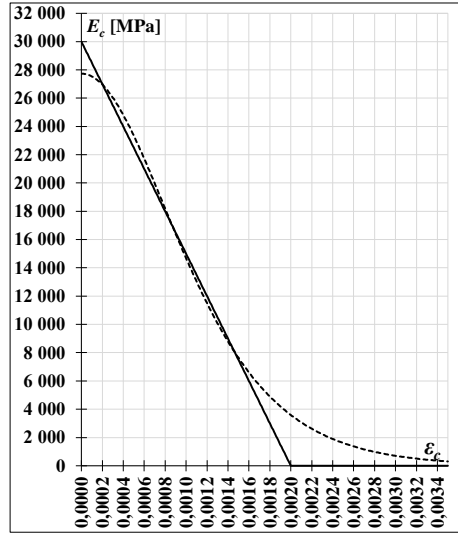


Figure 15. $E(\varepsilon)$ moduli for concrete

The σ - ε curves and $E(\varepsilon)$ moduli are according to Eurocode 2 (formulas (17), (18)) and formula (1)

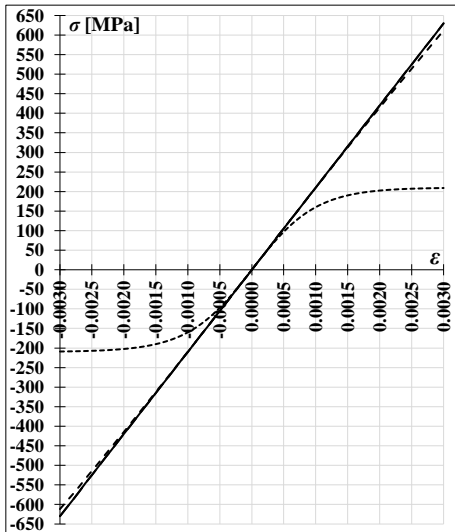


Fig. 16. Linear and nonlinear σ - ε relations

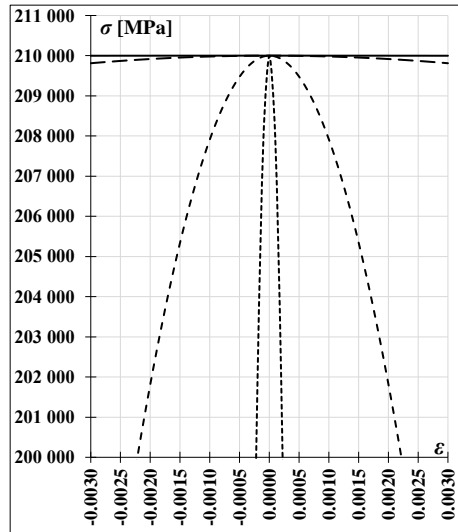


Fig. 17. Linear and nonlinear $E(\varepsilon)$ moduli

The nonlinear deformation modulus obtained by formula (18) is compared with the nonlinear deformation modulus set by formula (3), for σ - ε relation set by formula (1). The curves are shown in Fig. 15 as the continuous line is for formula (18):

In formulas (17) and (18), σ_c is the compressive stress in the concrete, ε_c is the compressive strain in the concrete, f_{cd} is the design value of the concrete compressive strength for concrete C 30/37, $\varepsilon_{c2} = 0,002$ is the strain for concrete C 30/37 at reaching the maximum (compressive) strength and $\varepsilon_{cu2} = 0,0035$ is the ultimate strain for concrete C 30/37 according to Table 3.1 of EUROCODE 2.

Also, with suitable, relatively large values of ε_{rul} (1; 0,1; 0,01) in formula (1), an "almost" perfectly elastic solid can be described. The stress-strain relationship is practically linear and becomes curvilinear only for values of ε_{rul} less than 0,01 (e.g., 0,001). The solid σ - ε line and the dashed σ - ε lines in Fig. 16 correspond to $\varepsilon_{rul} = 1$, $\varepsilon_{rul} = 0,1$ and $\varepsilon_{rul} = 0,01$ (the three lines almost coincide), and the dotted line σ - ε is for $\varepsilon_{rul} = 0,001$. Figure 17 shows the modulus of elasticity (in practice the Young's modulus for the first three cases) for the four values of ε_{rul} . In all cases, σ_{lim} is chosen such that the initial modulus of elasticity is 210 000 MPa.

In view of the discussion in Chapter 5, the Hooke's law in three dimensions (12) can be written by assuming a different nonlinear stress-strain relationship for each component of the Cauchy stress tensor and with one and the same Poisson's ratio in each direction:

$$\begin{aligned}\sigma_x &= \frac{1}{(1+\nu)(1-2\nu)} \cdot \left[(1-\nu) \cdot E(\varepsilon_{xx})_{xx} \cdot \varepsilon_{xx} + \nu \cdot E(\varepsilon_{xx})_{xx} \cdot \varepsilon_{yy} + \nu \cdot E(\varepsilon_{xx})_{xx} \cdot \varepsilon_{zz} \right]; \\ \sigma_y &= \frac{1}{(1+\nu)(1-2\nu)} \cdot \left[\nu \cdot E(\varepsilon_{yy})_{yy} \cdot \varepsilon_{xx} + (1-\nu) \cdot E(\varepsilon_{yy})_{yy} \cdot \varepsilon_{yy} + \nu \cdot E(\varepsilon_{yy})_{yy} \cdot \varepsilon_{zz} \right]; \\ \sigma_z &= \frac{1}{(1+\nu)(1-2\nu)} \cdot \left[\nu \cdot E(\varepsilon_{zz})_{zz} \cdot \varepsilon_{xx} + \nu \cdot E(\varepsilon_{zz})_{zz} \cdot \varepsilon_{yy} + (1-\nu) \cdot E(\varepsilon_{zz})_{zz} \cdot \varepsilon_{zz} \right];\end{aligned}\tag{19}$$

$$\tau_{yx} = E(\varepsilon_{yx})_{yx} \cdot \varepsilon_{yx} = \tau_{xy} = E(\varepsilon_{xy})_{xy} \cdot \varepsilon_{xy};$$

$$\tau_{zy} = E(\varepsilon_{zy})_{zy} \cdot \varepsilon_{zy} = \tau_{yz} = E(\varepsilon_{yz})_{yz} \cdot \varepsilon_{yz};$$

$$\tau_{xz} = E(\varepsilon_{xz})_{xz} \cdot \varepsilon_{xz} = \tau_{zx} = E(\varepsilon_{zx})_{zx} \cdot \varepsilon_{zx}.$$

Since $\varepsilon_{yx} = \varepsilon_{xy}$, $\varepsilon_{zy} = \varepsilon_{yz}$ and $\varepsilon_{xz} = \varepsilon_{zx}$, in order to satisfy the conditions of symmetry of the Cauchy stress tensor (the tensor of small strains) – mutuality of the tangential stresses, it is necessary to define the functions for the tangent moduli to be one and the same. The moduli $E(\varepsilon_{yx})_{yx} = E(\varepsilon_{xy})_{xy}$, $E(\varepsilon_{zy})_{zy} = E(\varepsilon_{yz})_{yz}$, and $E(\varepsilon_{xz})_{xz} = E(\varepsilon_{zx})_{zx}$ are the shear modulus and as it is accepted they are usually denoted by the letter "G" instead of by the letter "E". The dependence of the normal stresses on the linear strains in expressions (19) is obtained after inverting the [3×3] matrix formed by the coefficients (tangential moduli and the Poisson's ratio) of the relationship between the linear strains and the normal stresses in expressions (12).

In this case the corresponding element is the tangent modulus (the slope of the stress-strain curve at each point) $E(\varepsilon_{ij})_{ij}$ obtained after differentiation of the chosen analytical

dependence for the stresses from the strains. Then $E(\varepsilon_{ij})_{ij}$ is the modulus on the plane with normal in direction "i" along axis "j". Of course, as it is commonly accepted, we can denote with $E(\varepsilon_{ij})_i$ the modulus of the tension/compression along axis "i" (when "i"="j") and with $G(\varepsilon_{ij})_{ij}$ the shear modulus in direction "j" on the plane whose is the normal in direction "i" (when "i"≠"j"), e.g., [42]. A real orthotropic solid with a nonlinear stress-strain relation is thus defined. It has one or more Poisson's ratios corresponding to a contraction (elongation) in direction "j" when an elongation (contraction) is applied in direction "i". The only difference with the well-known stress-strain relationships is that they are defined for a perfectly elastic solid, e.g., [42], while using formulas (1), (13)÷(16) and other functions of this kind are defined for a solid taking into account the real nonlinear stress-strain relations. The Poisson's ratio can be one and the same or different in each direction.

6. Conclusions and future work

In the original methodology presented, various nonlinear stress-strain curves are defined. This allows many types of material nonlinearity of solids to be described in a very simple generalized way. The proposed formulation for describing the behavior of solids with material nonlinearity under [monotonic \(static\)](#) loading is not based on the different atomic, micro, meso- and macro structure of the solids. This formulation does not take into account the presence of any structural defects, neither point defects nor different types of dislocations. The only thing it takes into account is the existence of attractive and repulsive forces between elementary particles, atoms, molecules and larger structural units, which are common to all things in nature. This formulation corresponds to the physical principles that operate at all the levels in solids, namely that the forces of attraction and repulsion between elementary particles depend both on their charge and nonlinearly in practically all cases on the distance between them – the variation of this distance leads to a variation of the bond forces. All physical properties of different materials are described solely by the nonlinear stress-strain relationship. It alone defines the varied structure of different bodies, defects, and the rate and manner of application of mechanical stress, the rate of realization of stress and strain in solids. Each different structure and each different mode of stress application is described by a different stress-strain relationship selected appropriately for the case. The variation of stresses and strains due to temperature effect or change is not considered, besides the fact that the energy dissipated in the destruction of solids is generally taken as heat, without this heat influencing the process, but only as a consequence of it. The essence and purpose of the proposed formulation is to define a variety of stress-strain curves by means of relatively simple and uniform analytical functions. This is usually very useful in the numerical solution of material nonlinearity problems in solids and can greatly facilitate it in a number of cases. The latter, in turn, would probably make the use of material nonlinearity models of solids much more popular in engineering practice (and this seems to be desirable not only for Bulgaria but on a global scale [34]) and with much greater application in solving both static and dynamic, especially seismic, problems for as many different types of building structures as possible. Moreover, in the calculation of various practical problems, it is never superfluous to have a comparatively simple theoretical relation for the relationship between stresses and strains, even if it is not used every time.

The proposed theoretical formulation of the nonlinear relationship between stresses and strains in solids, based on hyperbolic functions, gives good agreement with a number of experimental results in various cases of material nonlinearity. Like Hooke's law, this relationship is based on the physical nature of the process, namely that deformations (the change of distances and angles between the elementary and not only the elementary particles of the body) along a given "generalized" direction cause stresses to arise only along this "generalized" direction as a consequence of the change in the interaction between the attractive and repulsive forces between the particles, e.g. [7, 45], etc. This change is, in fact, the stress that we want to determine. The fact that linear deformations along a given direction also cause linear deformations in the plane transverse to it is due to the principle of conservation of volume of all bodies, which is a manifestation of the Law of conservation of mass in cases where the density does not depend on time and does not change for any other reason, e.g., temperature change. The linear transversal strains, which cause the transversal stresses, do not change the nature of the phenomenon: "ut tensio, sic vis". In real solids, the σ - ε relationship is not linear, or is linear only for some types of solids and at relatively small deformations, and Poisson's ratio is mostly valid for the elastic region of the curve. As with Hooke's law, the proposed theoretical formulation of a nonlinear relationship between stresses and strains in solids defines the way in which the stress arises as a consequence of the strain caused by the load, but does not "create" this relationship as a product of the physical structure and chemical nature of the material. It is therefore suitable for the calculation of many problems, provided, however, that the actual failure of the solid is sufficiently approximated in the manner prescribed by the σ - ε relationship for each structural problem and corresponding load. Despite this drawback, a prescribed theoretical dependence of stresses on strains can greatly facilitate the numerical solution of material nonlinearity problems in solid mechanics. Such a simple and realistic dependence makes it easy to construct the tangent stiffness matrix for many different types of materials when using the regular or modified Newton-Raphson method and some quasi-Newton methods, and easy to calculate the actual nonlinear stresses (and corresponding reactive internal forces in FEM) in all cases.

The presented methodology could be very easily integrated into any commercial or other FEM program for solving problems of material nonlinearity in the field of solid mechanics. It can be applied to both 2D and 3D problems in the finite element method. This would make it relatively easy to refine the actual nonlinear behavior of both single structural elements and entire large structures. It would be particularly useful to do this not only for a static problem, but especially for dynamic and especially for seismic impact cases, where the seismic action acts as equivalent static forces. Using this compact and relatively simple, yet precise and versatile way to define material nonlinearity in solids can be a very fruitful approach in inelastic seismic pseudo static analysis, especially for the assessment of reinforced concrete structures. The latter is an interesting and important issue in seismic design as well as in general, e.g. [46 – 48], etc. For example, the application of this methodology in the Push-over analysis would make it relatively easy and with a high degree of reliability to estimate the "seismic" behavior factor in each different case, and thus more accurately evaluate the actual nonlinear behavior and seismic resistance of the structures.

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ОБОБЩЕНА ТЕОРЕТИЧНА ЗАВИСИМОСТ МЕЖДУ НАПРЕЖЕНИЯТА И ДЕФОРМАЦИИТЕ В ТВЪРДИ ТЕЛА С НЕЛИНЕЙНОСТ НА МАТЕРИАЛА В СЛУЧАЯ НА „МАЛКИ“ ДЕФОРМАЦИИ

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Ключови думи: нелинейност на материала, зависимост напрежение – деформация, хиперболични функции, „малки“ деформации, енергия на деформацията

РЕЗЮМЕ

Представен е оригинален теоретичен модел за формулиране на различни видове нелинейност на материала в пластични, крехки и хипереластични твърди тела в случай на „малки“ деформации. Различните зависимости напрежение – деформация са описани по обобщен, но много реалистичен, ясен и компактен начин. Представянето на аналитичната формула за тангенс хиперболичен чрез експоненти позволява само с един параметър за напреженията и няколко параметъра за деформациите да се опишат характерните криви напрежение – деформация за стоманите, различните криви на якост на натиск и опън за крехки материали, като бетон, както и специфичните криви за хипереластични материали. Тези параметри са естествените стойности на максималните напрежения и на граничните деформации, които възникват в твърдите тела при статично натоварване.

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