ГОДИШНИК НА УНИВЕРСИТЕТА ПО АРХИТЕКТУРА, СТРОИТЕЛСТВО И ГЕОДЕЗИЯ – СОФИЯ

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AN ALTERNATIVE APPROACH FOR MODE II STRAIN ENERGY RELEASE RATE DETERMINATION IN OVERHANGING BILAYERED POLYMER COMPOSITE BEAM

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Keywords: *unidirectional fiber reinforced polymer, mode II fracture, strain energy release rate, compliance technique*

Research area: *engineering mechanics, strength of materials*

ABSTRACT

The object under consideration is an overhanging beam made of two unidirectional fiber reinforced polymer composites. The beam is manufactured in the way that the fibers of the both composite materials are along the beam axis. A single crack situated parallel to the reinforcing fibers is supposed to exist between the layers. The loading conditions as well as crack location contributes to the Mode II fracture. The approach provided is based on the method of superposition according to which the loading scheme is divided into two simpler configurations. An expression for strain energy release rate is obtained applying the Fracture mechanics compliance technique.

1. Introduction

Fracture mechanics develops methods for determination of strain energy release rate, G, a parameter controlling crack propagation. First group of these methods relies on the change in strain energy due to the existence of crack in the structure while second group applies the compliance technique to obtain the result for G. It should be noted that both methods attack the problem directly, i.e. they investigate cracks in the given configuration of beam [1-6].

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However, the alternative methods for determination of fracture mechanics parameters controlling crack growth are needed either to facilitate the solution or to give another expression proving the validity of the results obtained. In the present paper, the approach based on principle of superposition is presented and the formula for strain energy release rate is derived.

The overhanging beam of rectangular cross-section is considered (Fig. 1). It has two layers each one of which is made of unidirectional fiber reinforced polymer composite situated in the manner that the fibers are parallel to the beam axis. A single crack exists between the layers in the middle of the cross-section. The load is a vertical force F applied at the overhang free end. Thus, the mode II fracture is provided.

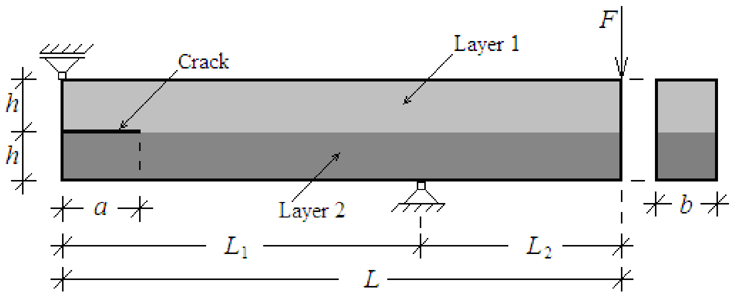

Fig. 1. Model of the overhanging beam investigated.

2. Determination of strain energy release rate

The formula for strain energy release rate, G, is obtained on the basis of compliance technique [7, 8]. According to this approach, G is:

$$G = \frac{F^2}{2b}\frac{dC}{da}, \qquad (1)$$

where $C = \frac{w}{F}$ is the compliance of the beam subjected to force F, while w is the vertical displacement of the point of application of F.

Here, the vertical displacement of the point of application of F is determined applying the principle of superposition according to which the loading scheme is resolved into two simpler configurations [9]. First scheme represents the un-cracked beam acted upon by force F. Second loading condition includes the axial loads $\overline{\tau}$ distributed uniformly on the two crack arms. It should be noted that in order to keep the equilibrium the sense of $\overline{\tau}$ applied to the upper crack arm is opposite to the sense of $\overline{\tau}$ applied to the lower crack arm (Fig. 2). Besides, the origin of x-axis coincides with the left beam section and the positive sense is to the right.

The beam compliance is obtained according to expression:

$$C = C_A + C_B, \qquad (2)$$

where C_A is the compliance of the un-cracked beam (loading configuration A), while C_B is the compliance of the beam loaded by axial distributed load $\overline{\tau}$ (loading configuration B).

Firstly, using Ref. [10], the modulus of elasticity of bilayered beam is obtained as a function of the moduli of elasticity of the layers, i.e.

$$E = \frac{E_1^2 + E_2^2 + 14E_1E_2}{8(E_1 + E_2)} . \tag{3}$$

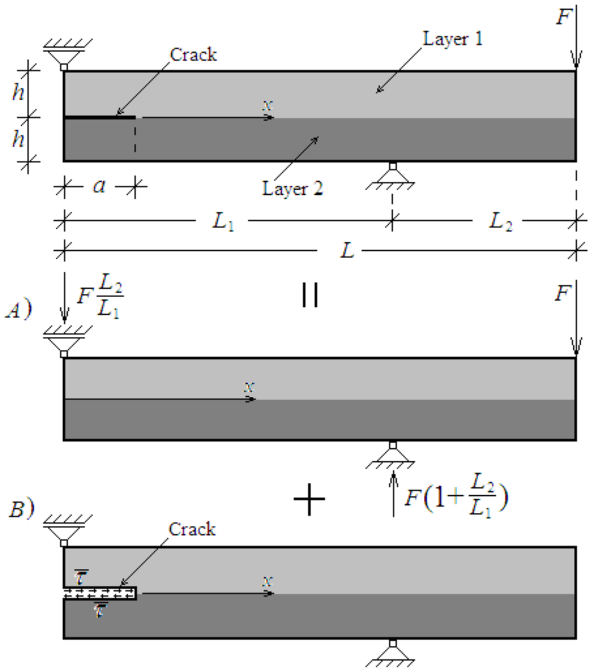

Fig. 2. Resolution of the loading scheme into two simpler configurations.

Further, the loading scheme *A* is considered and the vertical displacement of the point of application of force *F* is determined as:

$$w_A(L) = \frac{FL_2^2L}{3EI} = \frac{4FL_2^2L(E_1 + E_2)}{(E_1^2 + E_2^2 + 14E_1E_2)bh^3} . \tag{4}$$

Hence, the compliance of the un-cracked beam is:

$$C_A = \frac{w_A(L)}{F} = \frac{4L_2^2L(E_1 + E_2)}{(E_1^2 + E_2^2 + 14E_1E_2)bh^3} . \tag{5}$$

After that, the loading configuration B is analyzed and the vertical displacement of the point of application of the force F is determined. Here, due to the fact that the loading scheme is antisymmetric, the upper half of the beam is considered only. However, in order to satisfy such analysis, the equality of the moduli of elasticity of the two layers is assumed, i.e. $E = E_1 = E_2$. Of course, the statement that the moduli of elasticity of the two layers are different should be taken into account and it would be done when the deformations of the two crack arms are considered.

The magnitude of the distributed load applied to the upper crack arm is obtained using the well-known formula of Juravski as:

$$\overline{\tau} = \frac{VS}{bI} = \frac{3FL_2}{4bhL_1} . \tag{6}$$

Under the action of this distributed load, the shearing stresses τ of the sense opposite to the sense of $\overline{\tau}$ arise in the un-cracked portion of the beam (Fig.3).

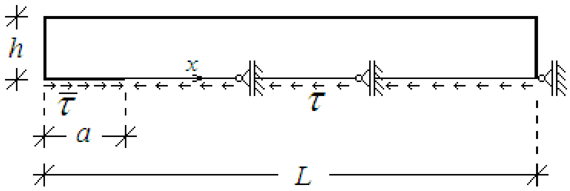

Fig. 3. Upper half of the beam.

Now, the task is to obtain the shearing stresses function because it relates to the functions of the beam deflection and slope. This is performed by means of Ref. [11] where the bilayered beam of different height and stiffness of the layers is examined. The following result is given for the shearing stresses function:

$$\tau(x) = C_1 e^{\lambda(x-a)} + C_2 e^{-\lambda(x-a)} . \tag{7}$$

In (7), C_1 and C_2 are the constants while λ is determined as:

$$\lambda = \sqrt{\xi\left(\frac{1}{E_1 bh} + \frac{1}{E_2 bh} + \frac{12h^2}{E_1 bh^3 + E_2 bh^3}\right)} = \sqrt{\frac{\xi\left(E_1^2 + E_2^2 + 14E_1E_2\right)}{E_1E_2(E_1+E_2)bh}} , \tag{8}$$

where ξ is coefficient accounting for the stiffness of the connection between the layers.

Further, taking into account the fact that the shearing stresses in the beam are small as magnitude, while the stiffness of the connection between the layers has great value, i.e. λ and $e^{\lambda(x-a)}$ are great numbers, it is assumed $C_1 = 0$. Thus, the first term in (7) is neglected and the shearing stresses function is obtained as:

$$\tau(x) = C_2 e^{-\lambda(x-a)} . \tag{9}$$

In order to find C_2, the equilibrium equation $\sum F_{ix} = 0$ is written and solved. The result is:

$$\frac{3FL_2}{4bhL_1} a + \frac{C_2 b}{\lambda}\left[e^{-\lambda(L-a)} - 1\right] = 0 . \tag{10}$$

Then, in accordance with condition $L >> a$, it is obtained $e^{-\lambda(L-a)} \approx 0$. Hence, the result derived for C_2 is:

$$C_2 = \frac{3FL_2\lambda a}{4bhL_1} . \tag{11}$$

Finally, the shearing stresses function is determined as:

$$\tau(x) = \frac{3FL_2\lambda a}{4bhL_1} e^{-\lambda(x-a)} . \quad (12)$$

Next step is to obtain the bending moment function in the un-cracked beam portion. In order to perform this, the infinitesimal segment of the beam is considered and equilibrium equation is written. The result is:

$$\frac{dM}{dx} = -\frac{h}{2} b\tau . \quad (13)$$

Then, substituting (12) into (13) the following result is found:

$$M(x) = -\frac{3FL_2 a}{8L_1} e^{-\lambda(x-a)} . \quad (14)$$

Further, the functions of the slope $\alpha_B(x)$ and deflection $w_B(x)$ of the un-cracked beam portion have to be obtained. For this purpose, the following relations are applied:

$$\alpha_B(x) = -\frac{1}{EI}\int M(x)dx = -\frac{9FL_2 a}{2EL_1 bh^3\lambda} e^{-\lambda(x-a)} + D_1 , \quad (15)$$

$$w_B(x) = \int \alpha(x)dx = \frac{9FL_2 a}{2EL_1 bh^3\lambda^2} e^{-\lambda(x-a)} + D_1 x + D_2 . \quad (16)$$

The solution continues with determination of the bending moment function in the beam cracked portion. The result is:

$$\bar{M}(x) = \int_0^x \left(-\frac{h}{2} b\bar{\tau} \right) dx = -\frac{bh}{2}\int_0^x \left(\frac{3FL_2}{4bhL_1} \right) dx = -\frac{3FL_2}{8L_1} x . \quad (17)$$

In order to take into account the different moduli of elasticity of two composites building the beam, the bending moment function is resolved into two components. According to Ref. [12], it is obtained:

$$\bar{M}_1(x) = -\frac{3}{8}\frac{E_1}{E_1+E_2}\frac{FL_2}{L_1} x , \; \bar{M}_2(x) = -\frac{3}{8}\frac{E_2}{E_1+E_2}\frac{FL_2}{L_1} x . \quad (18)$$

Then, expressions for slope and deflection in cracked beam portion are found as:

$$\bar{\alpha}_B(x) = \frac{9FL_2}{2(E_1+E_2)L_1 bh^3} x^2 + D_3 , \quad (19)$$

$$\bar{w}_B(x) = \frac{3FL_2}{2(E_1+E_2)L_1 bh^3} x^3 + D_3 x + D_4 . \quad (20)$$

Constants D_1, D_2, D_3, and D_4 are derived using the boundary conditions corresponding to the beam configuration. Then, since the purpose is to obtain the vertical displacement of the point of application of the force, the function $w_B(x)$ is written as:

$$w_B(x) = \frac{9FL_2a}{2EL_1bh^3\lambda^2}e^{-\lambda(x-a)} + \frac{4FL_2}{L_1^2bh^3}\left(\frac{a^3}{E_1+E_2} - \frac{a^3}{8E} + \frac{9a^2}{8E\lambda} + \frac{9a}{8E\lambda^2}\right)(x-L_1). \quad (21)$$

Further, $x = L$ is substituted into (21). The result is:

$$w_B(L) = \frac{4FL_2^2a^3}{(E_1+E_2)bh^3L_1^2} - \frac{4FL_2^2}{bh^3L_1^2}\left(\frac{E_1+E_2}{E_1^2+E_2^2+14E_1E_2}\right)\left(a^3 - \frac{9a^2}{8\lambda} - \frac{9a}{8\lambda^2}\right). \quad (22)$$

Thus, the compliance of the beam of loading configuration B is:

$$C_B = \frac{w_B(L)}{F} = \frac{4L_2^2a^3}{(E_1+E_2)bh^3L_1^2} - \frac{4L_2^2}{bh^3L_1^2}\left(\frac{E_1+E_2}{E_1^2+E_2^2+14E_1E_2}\right)\left(a^3 - \frac{9a^2}{8\lambda} - \frac{9a}{8\lambda^2}\right). \quad (23)$$

Then, substituting (6) and (28) into (2), it is obtained for C:

$$C = \frac{4L_2^2L(E_1+E_2)}{(E_1^2+E_2^2+14E_1E_2)bh^3} + \frac{4L_2^2a^3}{(E_1+E_2)bh^3L_1^2} -$$

$$-\frac{4L_2^2}{bh^3L_1^2}\left(\frac{E_1+E_2}{E_1^2+E_2^2+14E_1E_2}\right)\left(a^3 - \frac{9a^2}{8\lambda} - \frac{9a}{8\lambda^2}\right). \quad (24)$$

After that, the first order derivative of C with respect to a is derived as:

$$\frac{dC}{da} = \frac{12L_2^2a^2}{(E_1+E_2)bh^3L_1^2} - \frac{12L_2^2}{bh^3L_1^2}\left(\frac{E_1+E_2}{E_1^2+E_2^2+14E_1E_2}\right)\left(a^2 - \frac{3a}{4\lambda} - \frac{3}{8\lambda^2}\right). \quad (25)$$

Finally, (25) is introduced into (1). The expression for G takes the form:

$$G = \frac{6F^2L_2^2a^2}{(E_1+E_2)b^2h^3L_1^2} - \frac{6F^2L_2^2}{b^2h^3L_1^2}\left(\frac{E_1+E_2}{E_1^2+E_2^2+14E_1E_2}\right)\left(a^2 - \frac{3a}{4\lambda} - \frac{3}{8\lambda^2}\right). \quad (26)$$

3. Verification of the solution

In order to prove the validity of (26), the following formula from Ref. [13] is used:

$$G = \frac{72F^2L_2^2a^2}{b^2h^3L_1^2}\left[\frac{E_1E_2}{(E_1+E_2)(E_1^2+E_2^2+14E_1E_2)}\right] +$$

$$+\frac{kF^2L_2^2}{2bhL_1^2}\left[\frac{1}{(E_1+E_2)^2}\left(\frac{E_1^2}{G'_1} + \frac{E_2^2}{G'_2}\right) - \frac{G'_1+G'_2}{4G'_1G'_2}\right], \quad (27)$$

where the second term gives the shearing forces influence.

Firstly, formula (26) is rearranged, as follows:

$$G=\frac{72F^2L_2^2a^2}{b^2h^3L_1^2}\left[\frac{E_1E_2}{(E_1+E_2)(E_1^2+E_2^2+14E_1E_2)}\right]+$$
$$+\frac{9F^2L_2^2(2a\lambda+1)}{4b^2h^3L_1^2\lambda^2}\left(\frac{E_1+E_2}{E_1^2+E_2^2+14E_1E_2}\right). \tag{28}$$

Then, expressions (27) and (28) are compared. It should be noted that since the beam of big length is under consideration, the shearing forces influence can be neglected. Moreover, it has already been mentioned that the stiffness of the connection between layers has great value, and, thus $\frac{1}{\lambda^2}\approx 0$. Finally, the comparison reveals that the two formulas for G completely match.

4. Conclusions

The present paper deals with a single crack in overhanging bilayered composite beam. The beam is built up by two unidirectional fiber reinforced polymer composites and is loaded by a vertical force applied at the free end of the overhang beam part. The configuration proposed contributes to the mode II crack growth.

An alternative method for determination of strain energy release rate, G, is developed. It relies on fracture mechanics compliance technique, but the essence of the approach is that the loading scheme is resolved into two simpler configurations. Thus, the vertical displacement of the point of application of force F is easily obtained and substituted in the formula for beam compliance.

As a summary, it should be mentioned that the method developed in the article provides a different view on conventional problem for strain energy release rate determination. Applying the principle of superposition and mechanics of materials techniques, it is possible to derive result for G which can be used for comparison with well-known results in specialized literature.

LITERATURE

1. *Anderson T.L.* Fracture mechanics: Fundamentals and Applications, CRC Press, Boca Raton, USA, 1995

2. *Broek D.* Elementary Engineering Fracture Mechanics, Springer, 1986.

3. *Gospodinov G.* Introduction to Fracture Mechanics, www.uacg.bg/pw/gg/FM.

4. *Szekrenyes A.* "Prestressed composite specimen for mixed-mode I/II cracking in laminated materials", Journal of Reinforced Plastics and Composites, Vol. 29, pp. 3309-3321, 2010.

5. *Szekrenyes A.* "Interlaminar Fracture Analysis in the G_I-G_{II}-G_{III} Space Using Prestressed Transparent Composite Beams", Journal of Reinforced Plastics and Composites 30/19, pp. 1655-1669, 2011.

6. *Hutchinson J.W., Z. Suo*. “Mixed Mode Cracking in Layered Materials”, Advances in Applied Mechanics, Vol.29, pp. 63-191, 1992.
7. *Wang C. H.* Introduction to Fracture Mechanics, DSTO Aeronautical and Maritime Research Laboratory, Melbourne, 1996.
8. *Kosteas D.* Applied Fracture Mechanics, European Aluminium Association, 1994.
9. *Wang J., Qiao P.* “Novel Beam Analysis of the End Notched Flexure Specimen for Mode II Fracture”, Engineering Fracture Mechanics, Vol.71, pp. 219-231, 2004.
10. *Rzhanytsin A.* Built-up bars and plates, Stroyizdat, Moscow, 1986.
11. *Rzhanytsin A.* Engineering mechanics, Vizshaya shkola, Moscow, 1991.
12. *Kolev P., K. Mladenov.* Strength of materials, UACEG, Sofia, 1992.
13. *Rizov V., A. Mladensky*. „Strain Energy Release Rate Determination in the Case of Mode II Crack in Overhanging Bilayered Composite Beam”, Journal of Theoretical and Applied Mechanics, Vol.43, No.3, pp. 87-96, 2013.

АЛТЕРНАТИВЕН ПОДХОД ЗА ОПРЕДЕЛЯНЕ НА СКОРОСТТА НА ОСВОБОДЕНАТА ПОТЕНЦИАЛНА ЕНЕРГИЯ НА ДЕФОРМАЦИЯТА ПРИ ПУКНАТИНА ПО ОСНОВНА ФОРМА II В ДВУСЛОЙНА КОМПОЗИТНА ГРЕДА

А. Младенски[1]

Ключови думи: *еднопосочно армиран полимерен композитен материал, пукнатина по основна форма II, скорост на освободената потенциална енергия на деформацията, податливост*

Научна област: *строителна механика, съпротивление на материалите*

РЕЗЮМЕ

Обект на изследване в статията е проста греда с конзолно издаден край, изградена от два еднопосочно армирани полимерни композита. Гредата е произведена по такъв начин, че армировъчните влакна и на двата композита да са ориентирани по направление на нейната ос. В гредата предварително е направена пукнатина, която е успоредна на армировъчните влакна и е разположена между двата материала. Приетата товарна конфигурация и положението и вида на опорните устройства създават условия за нарастване на пукнатината по основна форма II. Приложен е подход за изследване на пукнатината, съгласно който товарната схема е разделена на две състояния. Формулата за скоростта на освободената потенциална енергия на деформацията е изведена съгласно концепцията за податливост в линейно-еластичната механика на разрушението.

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